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COUNTING THE NUMBER OF SUBGROUPS AND NORMAL SUBGROUPS OF THE GROUP U_{2np}, p IS AN ODD PRIME

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Abstract. The aim of this paper is to compute the number of subgroups and normal subgroups of the group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e, aba^{-1} = b^{-1}\rangle$, where p is an odd prime. Suppose $n = 2^r \prod_{1\le i\le s} p_i^{\alpha_i}$ in which p_i's are distinct odd primes, α_i's are positive integers and $t = \prod_{1\le i\le s} p_i^{\alpha_i}$. It is proved that the number of subgroups is $2\tau(2n) + (p-1)\left(\tau(\frac{n}{p}) + \tau(\frac{n}{2^r})\right)$, when $p \mid n$ and $2\tau(2n) + (p-1)\,[\tau(t)]$, otherwise. It will be also proved that this group has $\tau(2n) + \tau(n)$ normal subgroups.
Keywords. group; subgroup; dihedral group; finite group.

1. Introduction

Cavior [1] proved that the number of subgroups of a dihedral group of order $2n$ can by computed by $\tau(n)+\sigma(n)$. After publishing this work Calhoun [2] computed the number of subgroups in certain finite groups. For more information on this problem, we encourage the readers to consult the interesting book of Tărnăuceanu [6].

Following Darafsheh and Yaghoobian [3], we define:

$$U_{2nm} = \langle a, b \mid a^{2n} = b^m = e \mid aba^{-1} = b^{-1}\rangle.$$

This group has order $2nm$ and can be written as the semi-direct product of two cyclic groups that one of them is of order m and another one has order $2n$. Set $n = 2^r \prod_{1\le i\le s} p_i^{\alpha_i}$, where p_i's are distinct odd prime numbers and α_i's are positive integers. Shelash [4], introduced an algorithm for computing all subgroups and normal subgroups of a finite group. Shelash and Ashrafi [5] applied this algorithm to compute the number of minimal and maximal subgroups of certain finite groups.

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Here, we apply this algorithm to obtain the number of subgroups and normal subgroups of the group U_{2np}, where p is an odd prime.

The order table of U_{2np} is defined as the matrix $A = [a_{ij}]$ with $a_{ij} = 2^{i-1}c_{j-1}$, $1 \leq i \leq \tau(2^{r+1})$ and $1 \leq j \leq \tau(\prod_{1\leq i\leq s} p_i^{\alpha_i})$, where c_j is an odd divisor of $|U_{2np}|$ and the function $\tau(n)$ is defined as the number of positive divisors of n. For simplicity of our argument, we assume that $c_0 < c_1 < \cdots < c_{\alpha-1}$, where $\alpha = \tau(\prod_{1\leq i\leq s} p_i^{\alpha_i})$. For example if $|G| = 60$, then the order table of G is as follows:

a_{ij}	1	2	2^2
$c_0 = 1$	1	2	4
$c_1 = 3$	3	6	12
$c_2 = 5$	5	10	20
$c_3 = 15$	15	30	60

Throughout this paper our notations are standard and can be taken from the standard books on group theory. The function $\sigma(n)$ is defined as the summation of all divisors of n. Furthermore, the number of subgroups and normal subgroups of a group G are denoted by $Sub(G)$ and $NSub(G)$, respectively. Our calculations are done with the aid of GAP [7].

2. Main Results

The group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e \mid aba^{-1} = b^{-1}\rangle$ is a finite group of order $2np$, where p is an odd prime. Suppose $n = 2^r \prod_{1\leq i\leq s} p_i^{\alpha_i}$ in which p_i's are distinct odd primes and α_i's are positive integers. For simplicity of our argument, we assume that $t = \prod_{1\leq i\leq s} p_i^{\alpha_i}$. If $p = p_k \mid n$ then the order of U_{2np} is equal to $2^{r+1}p_1^{\alpha_1}\cdots p_k^{\alpha_{k+1}}\cdots p_s^{\alpha_s}$, otherwise it is $2^{r+1}p\prod_{1\leqslant i\leq s} p_i^{\alpha_i}$.

Lemma 2.1. *The following hold:*

1. *If q is even then $a^q b^w = b^w a^q$;*
2. *If q is odd then $a^q b^w = b^{-w} a^q$.*

Proof. By presentation of the group U_{2np}, we have $aba^{-1} = b^{-1}$ and so if q is even then $a^q b = ba^q$. Furthermore, if q is odd then $a^q b = b^{-1}a^q$. Choose positive integer w. Then $a^q b^w = ba^q b^{w-1}$. If q is even number, thus $a^q b^w = b^w a^q$. If q is odd number then $a^q b^w = b^{-1}a^q b^{w-1}$, then $a^q b^w = b^{-w}a^q$. □

Proposition 2.1. *Let $n = 2^r t$, $t = \prod_{1\leq i\leq s} p_i^{\alpha_i}$ and $m = p$ be an odd prime number. Then the structure description of the group U_{2np} is $C_t \times (C_p : C_{2^{r+1}})$.*

Proof. Suppose $\Phi = \langle a^{2^{r+1}}\rangle$, $\Psi = \langle b\rangle$ and $\Omega = \langle a^t\rangle$ are subgroups of U_{2np}. By Lemma 2.1, one can see that $g\Phi g^{-1} = g\langle a^{2^{r+1}}\rangle g^{-1} = \langle a^{2^{r+1}}\rangle = \Phi$, for all $g \in U_{2np}$.

Thus $\Phi \trianglelefteq U_{2np}$. Define $(\Psi : \Omega) = \langle b, a^t \rangle$. If i is odd then,

$$\begin{aligned} a^i b^j (\Psi : \Omega) b^{-j} a^{-i} &= a^i b^j \langle b, a^t \rangle b^{-j} a^{-i} \\ &= \langle a^i b^j b b^{-j} a^{-i}, a^i b^j a^t b^{-j} a^{-i} \rangle \\ &= \langle b, a^t b^{2j} \rangle \\ &= (\Psi : \Omega), \end{aligned}$$

and if i is an even number,

$$\begin{aligned} a^i b^j (\Psi : \Omega) b^{-j} a^{-i} &= a^i b^j \langle b, a^t \rangle b^{-j} a^{-i} \\ &= \langle a^i b^j b b^{-j} a^{-i}, a^i b^j a^t b^{-j} a^{-i} \rangle \\ &= \langle b, a^t b^2 \rangle \\ &= (\Psi : \Omega). \end{aligned}$$

Hence $(\Psi : \Omega)$ is a normal subgroup of U_{2np}. On the other hand, $\langle a^{2^{r+1}} \rangle \cap \langle b, a^t \rangle = e$ and $\frac{|\langle a^{2^{r+1}} \rangle| \times |\langle b, a^t \rangle|}{|\langle a^{2^{r+1}} \rangle \cap \langle b, a^t \rangle|} = 2np$, which completes our argument. $\square$

Lemma 2.2. *The group U_{2np} has the following types of subgroup:*

1. *The cyclic subgroups $\langle a^i \rangle$ of order $\frac{2n}{i}$, where $i \mid 2n$;*
2. *The subgroups $\langle a^i, b \rangle$ of order $\frac{2np}{i}$, where $i \mid 2n$;*
3. *The cyclic subgroups $\langle a^i b^j \rangle$, where $i \mid 2n$, $2p^k \nmid i$ and $j = 1, \cdots, p-1$.*

Proof. Set $H = \langle a^i \rangle$ and $K = \langle b \rangle$, $i \mid 2n$. By presentation of U_{2np}, K is normal and so $HK = \langle a^i, b \rangle$ has order $\frac{2np}{i}$. The result now follows from Lemma 2.1. $\square$

Proposition 2.2. *Let $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ be a positive integer and p be an odd prime number. The following hold:*

1. *There is at most one subgroup of order k such that $2 \mid k, 2^{r+1} \nmid k$ and $p \nmid k$;*
2. *If $p \mid n$, then there exists one subgroup of order k such that $p^{\alpha_i+1} \mid k$;*
3. *There exists p subgroups of order k when $p \nmid k$ and $2^{r+1} \mid k$;*
4. *There exists $\sigma(p)$ subgroups of order k when $p \mid k$ and $p^{\alpha_{i+1}} \nmid k$.*

Proof. Our main proof will consider the following parts:

1. Suppose $p \nmid 2^h v$, $1 \leq h \leq r$, and $v \mid n$. Then $\langle a^{\frac{2^{r+1-h_t}}{v}} \rangle$ is a cyclic group of order $2^h v$ and the order of subgroups $\langle a^{\frac{2^{r+1-h_m}}{v}} b \rangle$ and $\langle a^{\frac{2^{r+1-h_m}}{v}}, b \rangle$ are not $2^h v$. We now apply Lemma 2.2 to get the result.

2. Suppose $2^{r+1} \mid k$. Since $\frac{t}{v}$ is an odd number, by Lemma 2.1 $\langle a^{\frac{t}{v}} b^j \rangle$ are cyclic subgroups of order $2^{r+1}v$, $1 \le j \le p$.

3. Consider the subgroups $\langle a^{\frac{2n}{2^h p}} \rangle$ and $\langle a^{\frac{2n}{2^h p}}, b \rangle$, where $1 \le h \le r+1$. Since there are $p-1$ subgroups of type $\langle a^{\frac{2n}{2^h p}} b^j \rangle$, $1 \le j \le p-1$, the number of all subgroups of order k is equal to $\sigma(p)$

Hence the result. □

Theorem 2.1. *Let p be an odd prime and $n = 2^r \prod_{1 \le i \le s} p_i^{\alpha_i}$, where p_i's are distinct odd primes, α_i's are positive integers and $t = \prod_{1 \le i \le s} p_i^{\alpha_i}$. Then the number of all subgroups of the group U_{2np} is given by the following:*

1. *If $p \mid n$ then $Sub(U_{2np}) = 2\tau(2n) + (p-1)\left[\tau(\frac{n}{p}) + \tau(\frac{n}{2^r})\right]$.*

2. *If $p \nmid n$ then $Sub(U_{2np}) = 2\tau(2n) + (p-1)\left[\tau(t)\right]$.*

Proof. By presentation of the group U_{2np}, it has $\tau(2n)$ subgroups contained in $\langle a \rangle$. Since $\langle b \rangle$ is a normal subgroup, the group U_{2np} has $\tau(2n)$ subgroups of the form $H\langle b \rangle$ such that H is a subgroup of $\langle a \rangle$. We now assume that $p \mid n$. By Lemma 2.2, it is enough to count the number of subgroups in the form $\langle a^i b^j \rangle$, where $i \mid 2n$, $2p^\alpha \nmid i$ and $1 \le j \le p-1$. Note that $2n$ has exactly $\tau(\frac{2n}{2^{r+1}}) = \tau(\frac{n}{2^r})$ odd divisors and the number of all divisors of $2n$ such that $2p \mid i$ and $2p^\alpha \nmid i$ is equal to $\tau(\frac{2n}{2p}) = \tau(\frac{n}{p})$. So the group U_{2np} has exactly $(p-1)[\tau(\frac{n}{p}) + \tau(\frac{n}{2^r})]$ subgroups, when $p \mid n$. If $p \nmid n$, then the number of subgroups of type $\langle a^i b^j \rangle$ is equal to $(p-1)\tau(\frac{n}{2^r}) = (p-1)\tau(t)$. □

We are now ready to count the number of normal subgroups of the group U_{2np}.

Lemma 2.3. *The normal subgroup of the group U_{2np} has one of the following forms:*

1. *All cyclic subgroups $\langle a^i \rangle$ such that $2 \mid i \mid 2n$;*

2. *All subgroups $\langle a^i, b \rangle$, when $i \mid 2n$.*

Proof. The first part follows from Lemma 2.1. We apply the presentation of U_{2np} to prove that $\langle a^k, b \rangle$ is normal, when $k \mid 2n$. Choose the element $a^i b^j$ in U_{2np}. Then we have four cases for the subgroup $a^i b^j \langle a^k, b \rangle b^{-j} a^{-i}$ as follows:

1. *k and i are even numbers.* In this case $\langle a^i b^j a^k b^{-j} a^{-i}, a^i b^j b b^{-j} a^{-i} \rangle = \langle a^k, b \rangle$, as desired.

2. *k is even and i is odd.* Then, $\langle a^i b^j a^k b^{-j} a^{-i}, a^i b^j b b^{-j} a^{-i} \rangle = \langle a^k, b \rangle$ which proves our claim.

3. *k and i are odd numbers.* This shows that $\langle a^ib^ja^kb^{-j}a^{-i}, a^ib^jbb^{-j}a^{-i}\rangle = \langle a^kb^{2j}, b\rangle = \langle a^k, b\rangle$.
4. *k is even and i is odd.* In this case, $\langle a^ib^ja^kb^{-j}a^{-i}, a^ib^jbb^{-j}a^{-i}\rangle = \langle a^kb^{-2j}, b\rangle = \langle a^k, b\rangle$.

Note that a^k and a^kb^j has the same order, when k is odd number. □

Choose $a^i \in U_{2np}$, where i is an odd number. Then $a^i\langle a^ib^j\rangle a^{-i} = \langle a^ia^ib^ja^{-i}\rangle = \langle a^ib^{-j}\rangle$. Since $\langle a^ib^{-j}\rangle \neq \langle a^ib^j\rangle$, all subgroups $\langle a^ib^j\rangle$, $1 \leq j \leq p$ and $i \mid 2n$, are not normal in U_{2np}.

Theorem 2.2. *The number of normal subgroups in the group U_{2np} is given by $NSub(U_{2np}) = \tau(2n) + \tau(n)$.*

Proof. Let p be an odd prime and $n = 2^r\prod_{1\leq i\leq s} p_i^{\alpha_i}$, where p_i's are distinct odd primes, α_i's are positive integers and $t = \prod_{1\leq i\leq s} p_i^{\alpha_i}$. To prove the theorem, we apply Lemma 2.3. We now that each subgroup of type $\langle a^i\rangle$, i is even, is normal. Since

$$\begin{aligned}
\tau(2^{r+1}t) - \tau(t) &= \\
\tau(2^{r+1}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) &= (r+2)\tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) \\
&= (r+1)\tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) \\
&= \tau(2^rp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) \\
&= \tau(n),
\end{aligned}$$

$\tau(2^{r+1}t)$ is the number all divisors of $2n$ and $\tau(t)$ is the number of odd divisors of $2n$, $\tau(2^{r+1}t) - \tau(t) = \tau(2^rt) = \tau(n)$ is the number of even divisors of $2n$. On the other hand, the number of all normal subgroups of type $\langle a^i, b\rangle$, $i \mid 2n$, is equal to $\tau(2n)$. Therefore, $NSub(U_{2np}) = \tau(2n) + \tau(n)$. □

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